Lecture 14
Introduction to Deep Unsupervised Learning

CMSC 35246: Deep Learning

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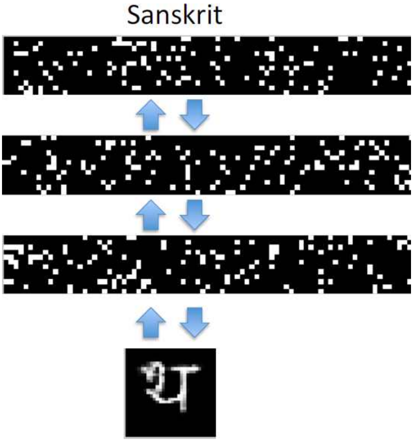
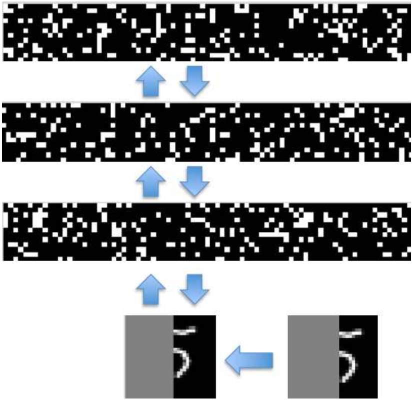
- $P(X)$ is defined in terms of $P(X|Y)$ or the best model of X (unsupervised learning) must involve the labels Y as a latent factor
- The idea of representation learning is to uncover the *latent variables* that *explain* X

Unsupervised Learning



Unsupervised Learning



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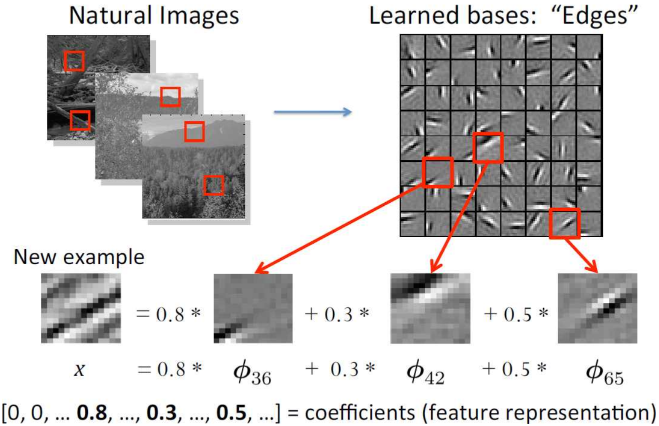
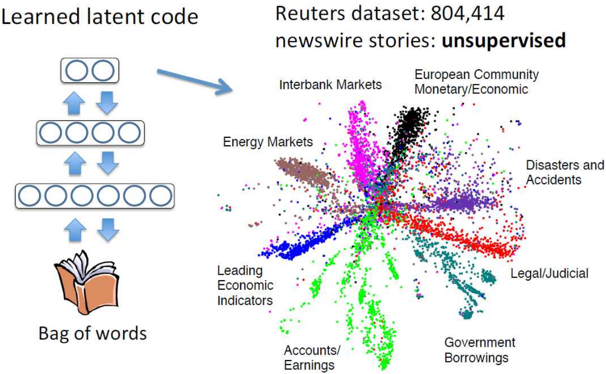


Slide credit: Honglak Lee

Unsupervised Learning



G. Hinton and R. Salakhutdinov, "Semantic Hashing", 2006

Unsupervised Learning

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- Learning features for classification
- Semi-supervised learning

Unsupervised Deep Learning

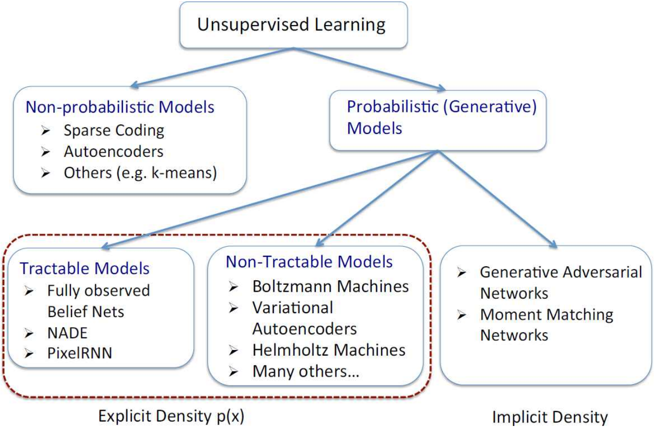

Figure: Ruslan Salakhutdinov

Warm Up

Linear Projections

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- Which is just:

$$Error = \sum_{i=1}^{N} \sum_{j=p+1}^{N} \mathbf{h}_j^T \mathbf{x}_i \mathbf{x}_i^T \mathbf{h}_j$$

Linear Projections

$$Error = \sum_{j=p+1}^{N} \mathbf{h}_j^T \Sigma \mathbf{h}_j$$

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$$\min_{\mathbf{u}} \mathbf{u}\Sigma\mathbf{u} + \lambda(1 - \mathbf{u}^T\mathbf{u})$$

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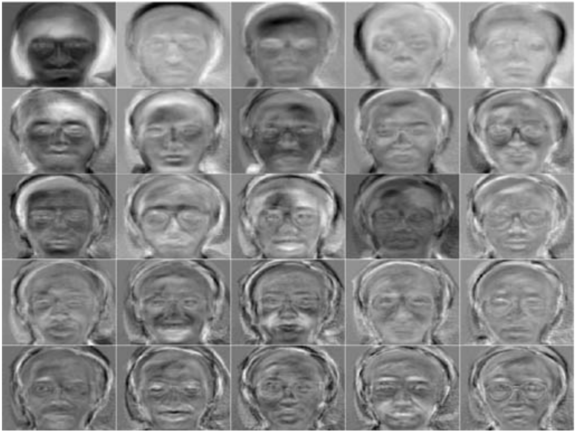
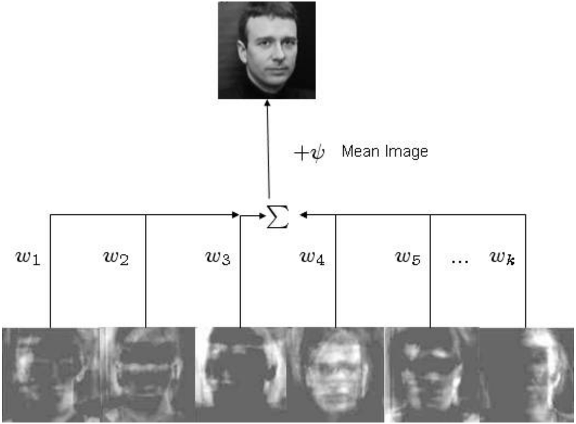
- Solutions are eigenvectors!

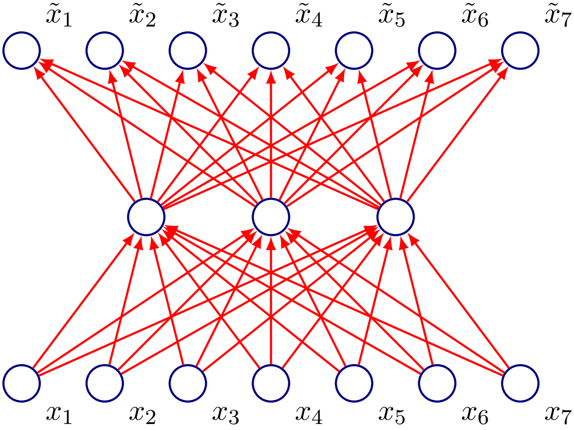
PCA on Face Images: Eigenfaces



Eigenfaces: Features



A Linear Neural Network



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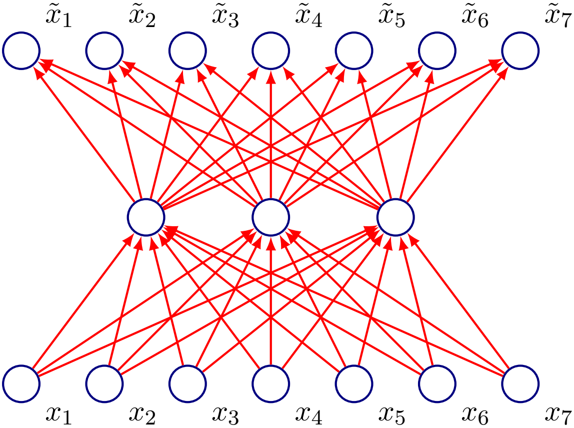
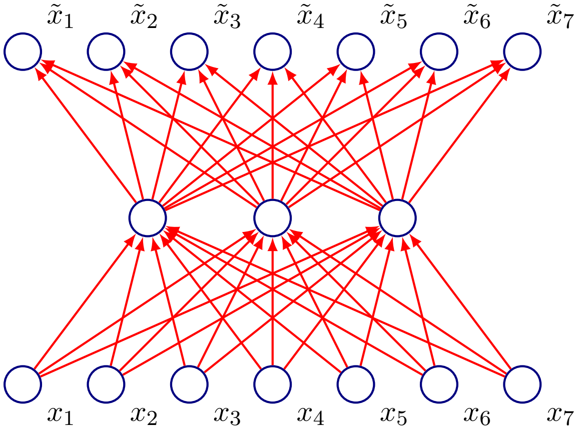


- Encoding: $\mathbf{x} \to \mathbf{h} = W\mathbf{x}$. Decoding: $\mathbf{h} \to \tilde{\mathbf{x}} = V\mathbf{h}$

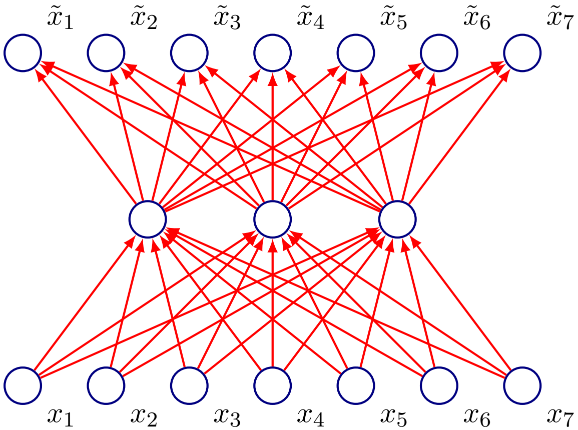
A Linear Neural Network



- Objective:

$$\min_{W,V} \|\mathbf{x} - VW\mathbf{x}\|_2^2$$

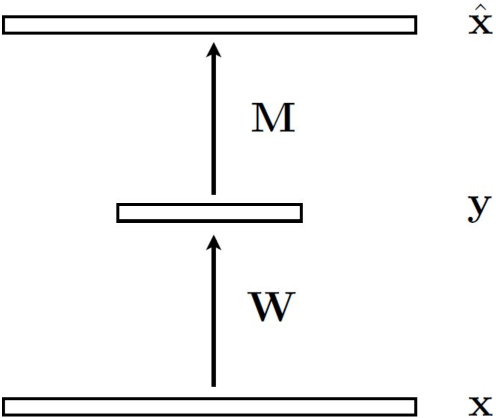
A Linear Neural Network



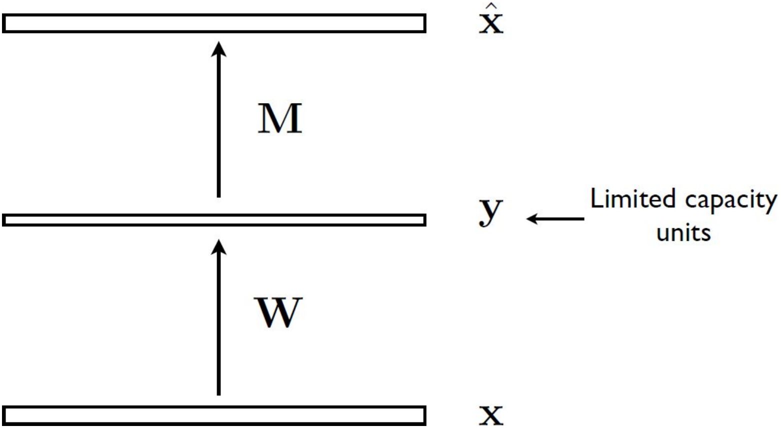
- This is a linear Autoencoder

Autoencoder: Non-Linear PCA

$$\min_{\mathbf{W},\mathbf{M}} |\mathbf{x} - \hat{\mathbf{x}}|^2$$

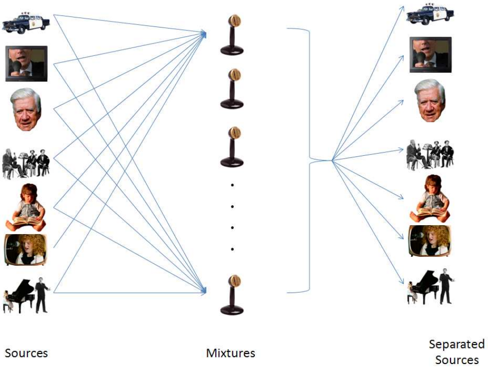


Autoencoder: Implicit Bottleneck



Another Linear Model: ICA

- Canonical example: Cocktail party problem



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- Here the bases are *independent* of each other

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- Some PCs are more important than others, not in the case with ICA

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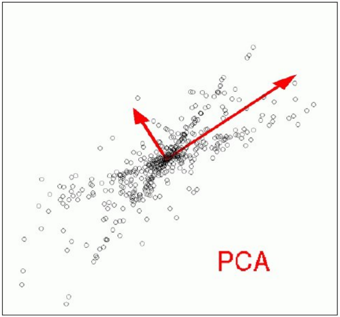
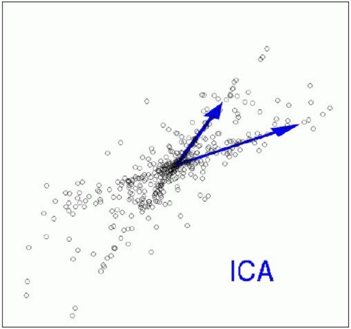
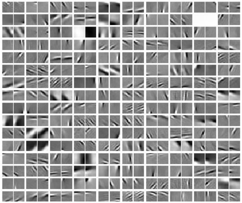




Filters



Sparse Coding

- Objective: Given a set of input vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$, learn a dictionary of bases $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_p$ such that:

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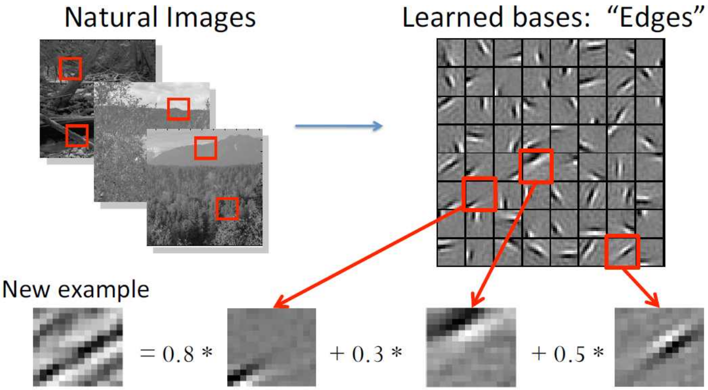
- Like before, but data is now a *sparse* linear combination of bases

Sparse Coding



$$x = 0.8 * \phi_{36} + 0.3 * \phi_{42} + 0.5 * \phi_{65}$$

[0, 0, ... **0.8**, ..., **0.3**, ..., **0.5**, ...] = coefficients (feature representation)

Sparse Coding

- Optimization: Given $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N \in \mathbb{R}^d$, learn dictionary $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_p \in \mathbb{R}^d$ (arranged as $H = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_p] \in \mathbb{R}^{d \times p}$) such that:

$$\min_{\mathbf{a}_1, \ldots, \mathbf{a}_N, H} \sum_{i=1}^{N} \|\mathbf{x}_i - H\mathbf{a}_i\|_2^2 + \lambda \sum_{i=1}^{N} \|\mathbf{a}_i\|_1$$

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- Sparsity term: $\|\mathbf{a}_i\|_1$

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  2. Fix bases $H = [\mathbf{h}_1 \ldots, \mathbf{h}_p]$ and optimize for codes $\mathbf{a}_i$

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  2. Fix bases $H = [\mathbf{h}_1 \ldots, \mathbf{h}_p]$ and optimize for codes $\mathbf{a}_i$
  3. Fix codes $\mathbf{a}_i$ and optimize for H (convex)

Sparse Coding:Test Time

- Given a new patch $\tilde{\mathbf{x}} \in \mathbb{R}^d$ and learned dictionary $H = [\mathbf{h}_1 \ldots, \mathbf{h}_p]$, we find the code $\tilde{\mathbf{a}}$ as:

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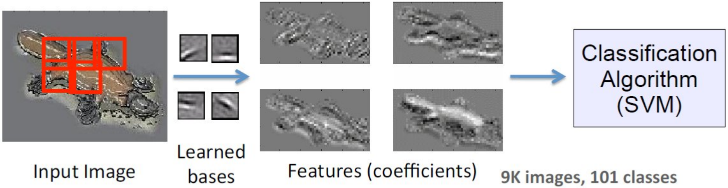
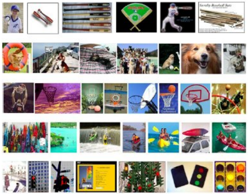
- $\tilde{\mathbf{a}}$ will be a sparse representation for $\tilde{\mathbf{x}}$

Image Classification

Evaluated on Caltech101 object category dataset.



9K images, 101 classes



Algorithm	Accuracy
Baseline (Fei-Fei et al., 2004)	16%
PCA	37%
Sparse Coding	**47%**

Slide Credit: Honglak Lee

Lee, Battle, Raina, Ng, 2006

Features for Faces

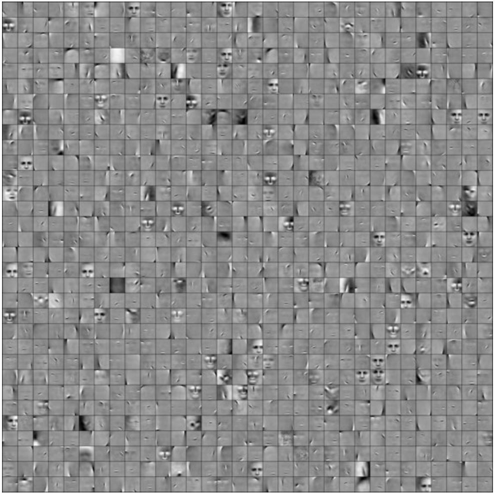


Figure: Charles Cadieu

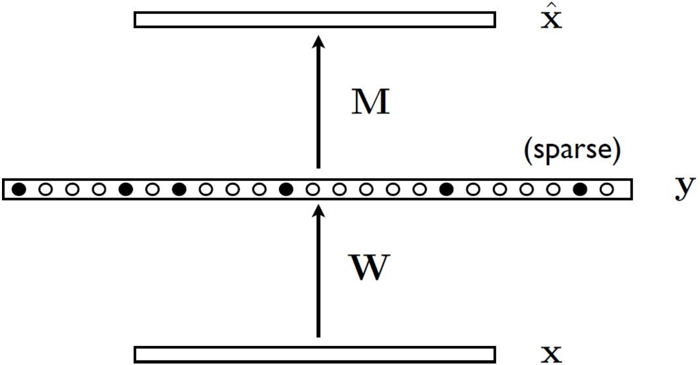
Encoding-Decoding

- Encoding: Implicit non-linear (in $\mathbf{x}$) encoding
- Decoding: Explicit linear decoding
- Can be overcomplete

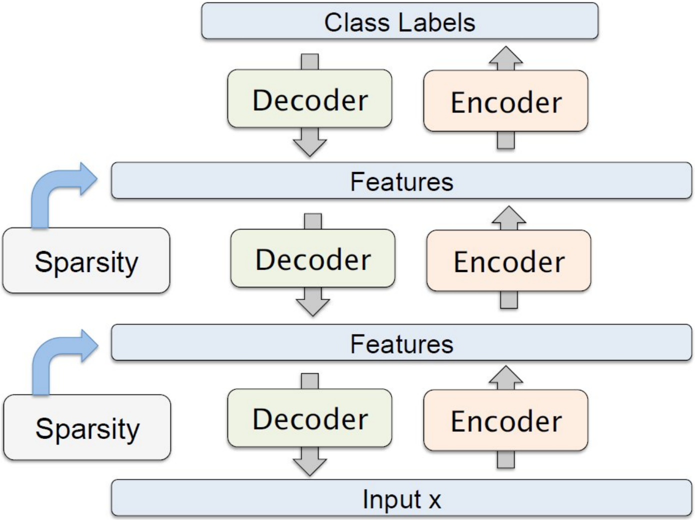
Encoding-Decoding

Simple Neural Network

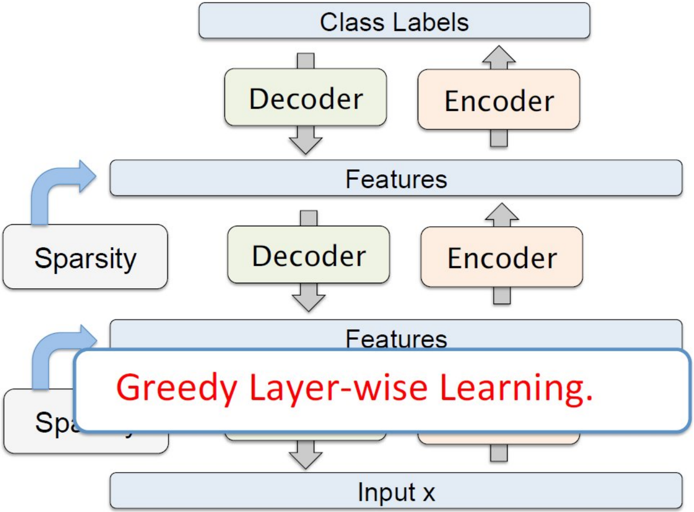
Sparse Autoencoders



Stacked Autoencoders



Pre-Training



Deep Autoencoders (2006)

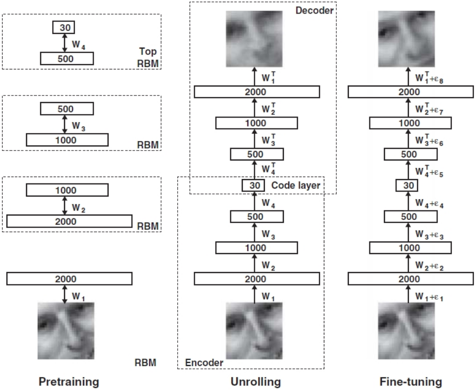
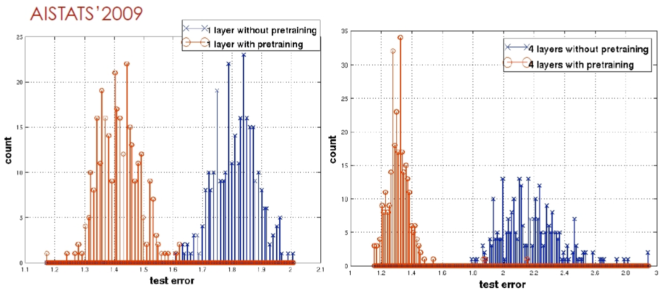


Fig. 1. Pretraining consists of learning a stack of restricted Boltzmann machines (RBMs), each having only one layer of feature detectors. The learned feature activations of one RBM are used as the "data" for training the next RBM in the stack. After the pretraining, the RBMs are "unrolled" to create a deep autoencoder, which is then fine-tuned using backpropagation of error derivatives.

G. E. Hinton, R. R. Salakhutdinov, Reducing the dimensionality of data with neural networks, Science, 2006

It was hard to train deep feedforward networks from scratch in 2006!

Effect of Unsupervised Pre-training



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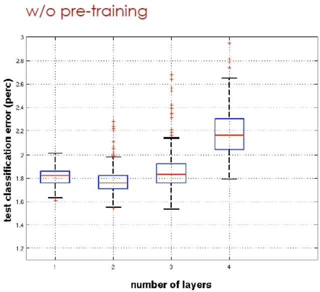

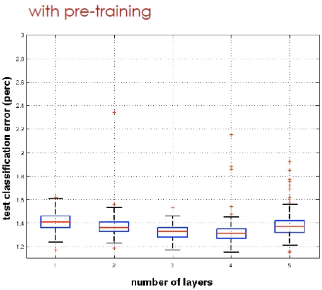

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- Optimization: Unsupervised pre-training leads to better regions of the space i.e. better than random initialization

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- De-noising Autoencoders: Input is corrupted by noise, but we attempt to reconstruct the uncorrupted image
- Contractive Autoencoders: The regularization term penalizes for the derivative:

$$\Omega(\mathbf{h}, \mathbf{x}) = \lambda \sum_i \|\nabla_{\mathbf{x}} \mathbf{h}_i\|_2^2$$

De-Noising Autoencoder: Intuition

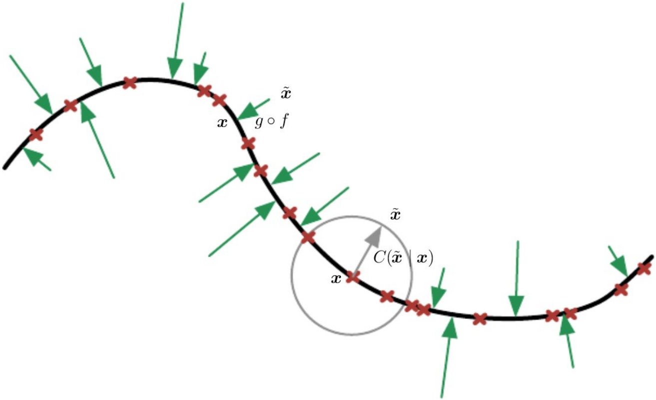

Figure: Goodfellow *et al.*

Back to Simple Models

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$$\mathbf{x} \sim \mathcal{N}(\mathbf{x}; b, WW^T + \Sigma)$$

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- Approaches PCA as $\sigma \to 0$

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- Energies are in the log-probability domain:

$$\text{Energy}(\mathbf{x}) = \log \frac{1}{(ZP(\mathbf{x}))}$$

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- Contrast this with mixture models

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- Free Energy is just a marginalization of energies in the log-domain:

$$\textsf{FreeEnergy}(\mathbf{x}) = -\log \sum_{\mathbf{h}} \exp^{-(\textsf{Energy}(\mathbf{x},\mathbf{h}))}$$