Lecture 15
Introduction to Deep Unsupervised Learning II

CMSC 35246: Deep Learning

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Recap: Unsupervised Deep Learning

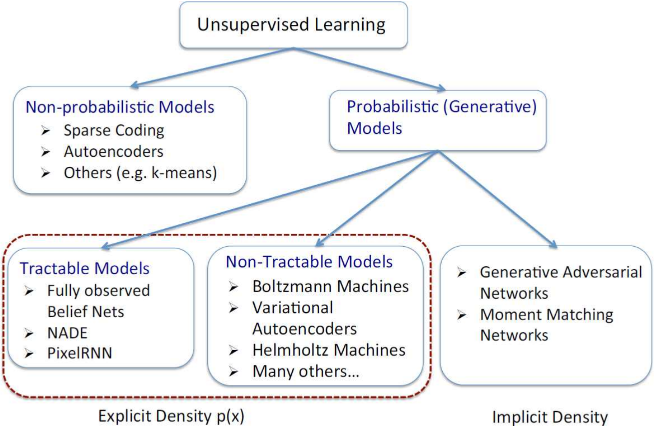

Figure: Ruslan Salakhutdinov

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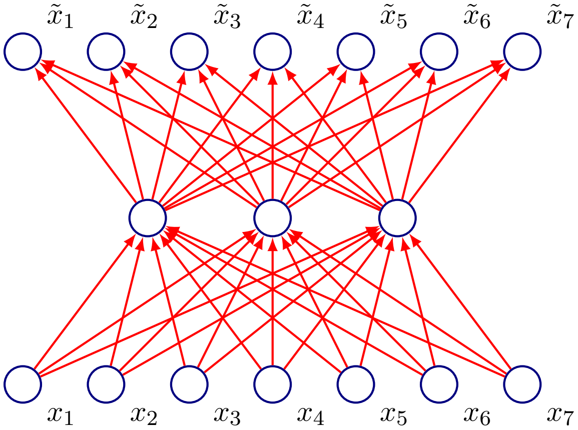
- The bases are orthogonal. Coefficient vectors are dense

Recap: Linear Autoencoder



- Encoding: $\mathbf{x} \rightarrow \mathbf{h} = W\mathbf{x}$. Decoding: $\mathbf{h} \rightarrow \tilde{\mathbf{x}} = V\mathbf{h}$

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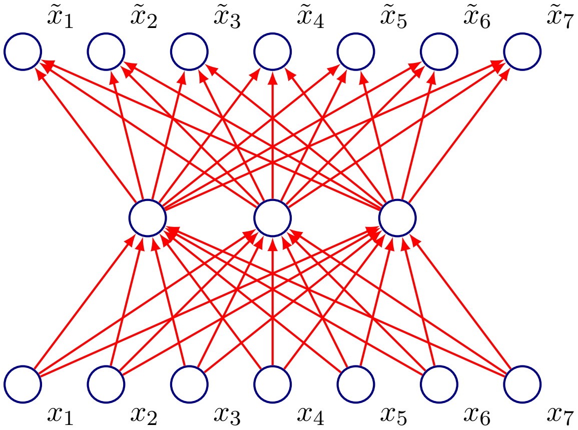

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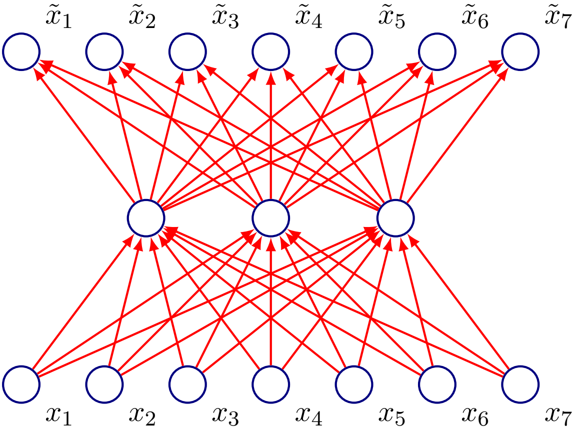
Recap: Simple Non-Linear Autoencoder



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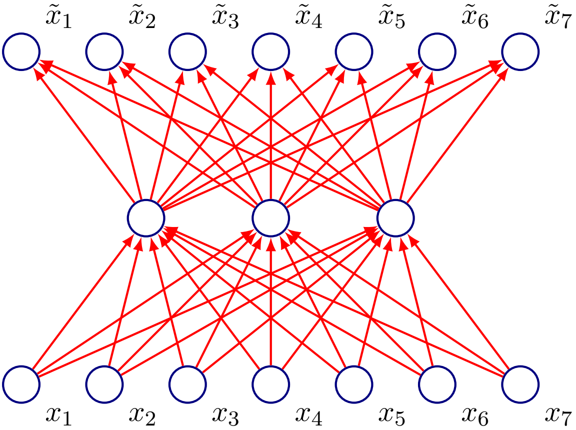

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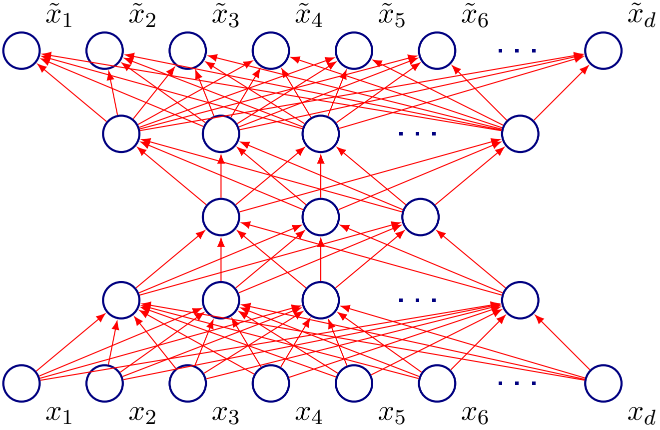

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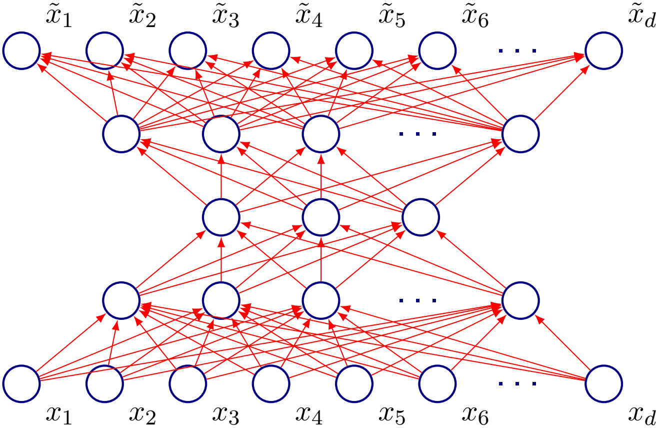

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- The forms for encoding and decoding can be different than those specified
- We get non-linear projections or *representations* of the data
- Can be seen as a form of non-linear PCA

Recap: Sparse Coding

- Objective: Given a set of input vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$, learn a dictionary of bases $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_p$ such that:

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- In PCA, the bases $\mathbf{h}$'s were orthogonal and the codes for the $\mathbf{x}$ i.e. $\mathbf{a}$'s were dense.
- Here, the bases need not be orthogonal, but the codes are sparse

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- Optimization Problem: Given $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N \in \mathbb{R}^d$, learn dictionary $\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_p \in \mathbb{R}^d$ (arranged as $H = [\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_p] \in \mathbb{R}^{d \times p}$) such that:

$$\min_{\mathbf{a}_1,\ldots,\mathbf{a}_N, H} \sum_{i=1}^{N} \|\mathbf{x}_i - H\mathbf{a}_i\|_2^2 + \lambda \sum_{i=1}^{N} \|\mathbf{a}_i\|_1$$

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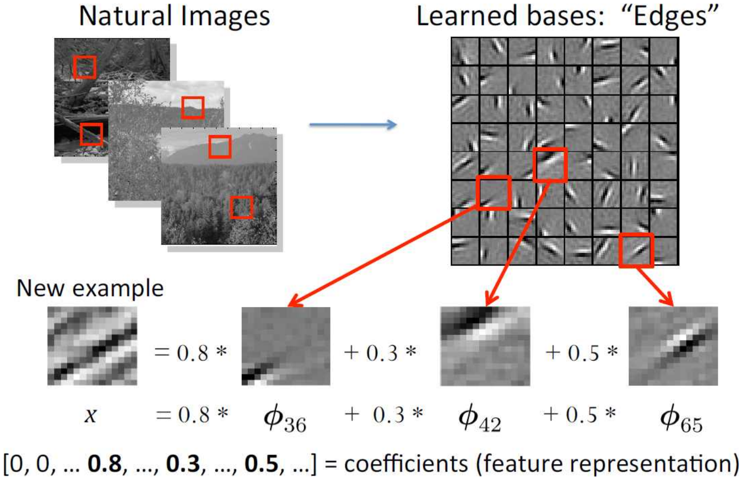
- Sparsity term: $\|\mathbf{a}_i\|_1$

Sparse Coding



The ϕ's here are our $\mathbf{h}$
Figure: Honglak Lee

Sparse Coding:Test Time

- Given a new patch $\tilde{\mathbf{x}} \in \mathbb{R}^d$ and learned dictionary $H = [\mathbf{h}_1 \ldots, \mathbf{h}_p]$, we find the code $\tilde{\mathbf{a}}$ as:

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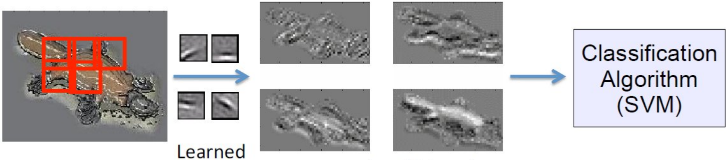
- Again, $\tilde{\mathbf{a}}$ is our representation or *code* for $\tilde{\mathbf{x}}$ that we can use as features for classification

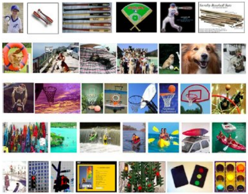
Image Classification

Evaluated on Caltech101 object category dataset.



Input Image — Learned bases — Features (coefficients)

9K images, 101 classes



Algorithm	Accuracy
Baseline (Fei-Fei et al., 2004)	16%
PCA	37%
Sparse Coding	**47%**

Slide Credit: Honglak Lee

Lee, Battle, Raina, Ng, 2006

Features for Faces

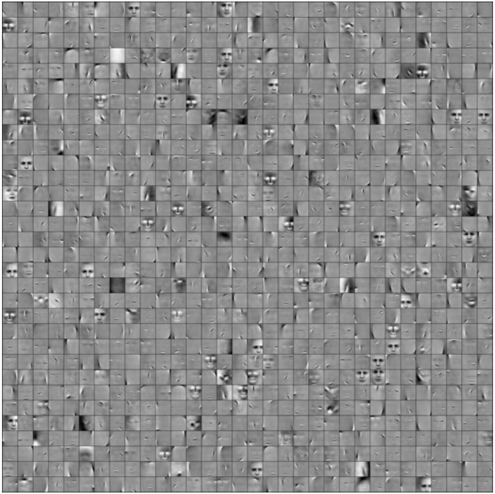


Figure: Charles Cadieu

Encoding-Decoding

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- Bases is overcomplete
- In PCA, plain autoencoders (i.e. Non-Linear PCA) overcomplete representations don't make much sense (can just copy input!)

- Like before, as in the case of PCA, let us try to write sparse coding as a neural network

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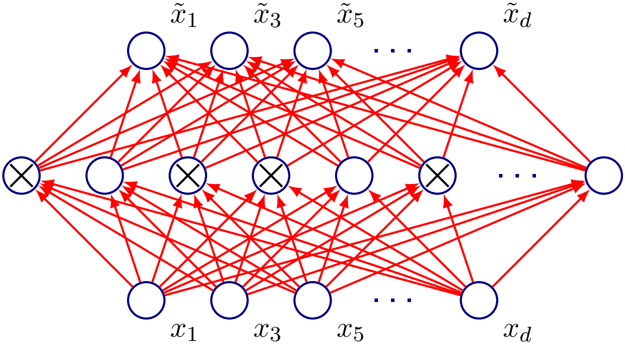
- Will lead to another kind of auto-encoder

Implicit Bottleneck



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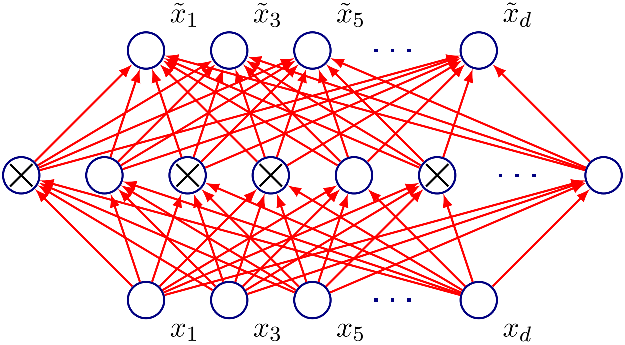
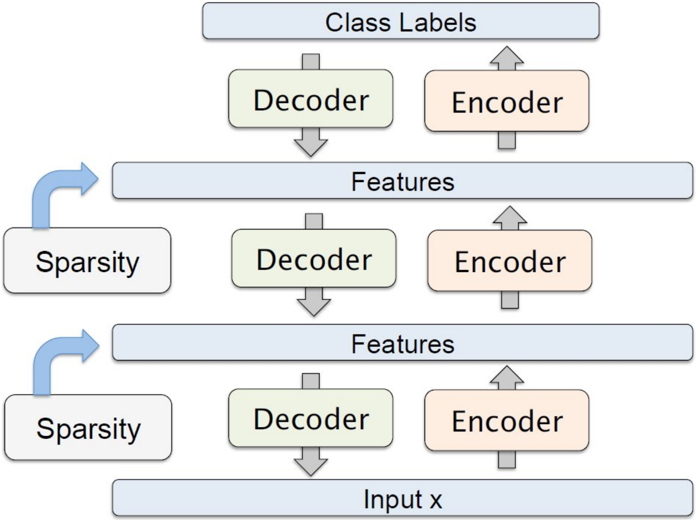
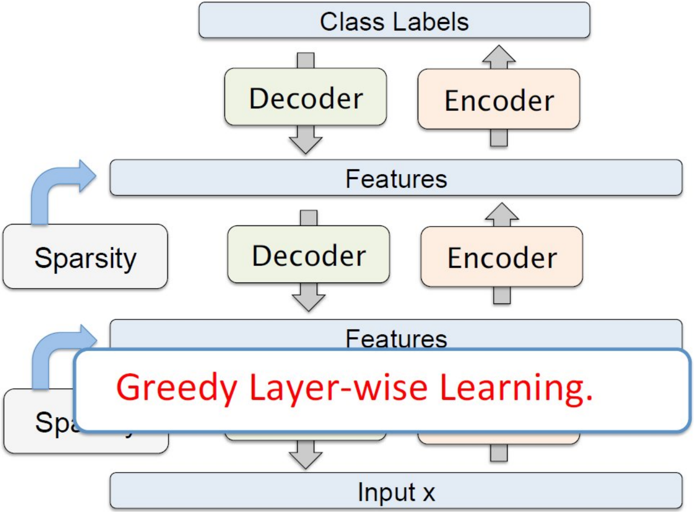


- Encoding: $\mathbf{h} = \tanh(W\mathbf{x})$. Decoding: $\tilde{\mathbf{x}} = V\mathbf{h}$
- PS: Modified model than the sparse coding model we saw (but to emphasize nonlinearity in encoding, and linear decoding)

Stacked Autoencoders



Pre-Training



Deep Autoencoders (2006)

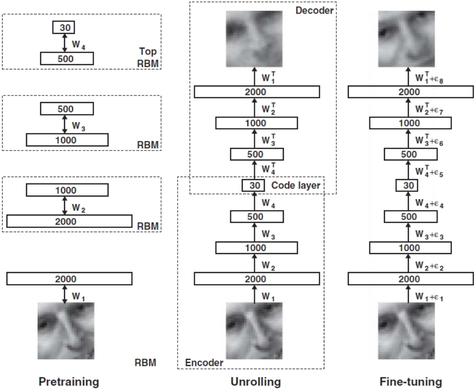
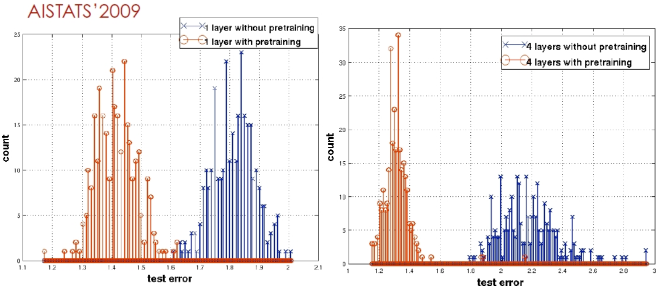


Fig. 1. Pretraining consists of learning a stack of restricted Boltzmann machines (RBMs), each having only one layer of feature detectors. The learned feature activations of one RBM are used as the "data" for training the next RBM in the stack. After the pretraining, the RBMs are "unrolled" to create a deep autoencoder, which is then fine-tuned using backpropagation of error derivatives.

G. E. Hinton, R. R. Salakhutdinov, Reducing the dimensionality of data with neural networks, Science, 2006

It was hard to train deep feedforward networks from scratch in 2006!

Effect of Unsupervised Pre-training



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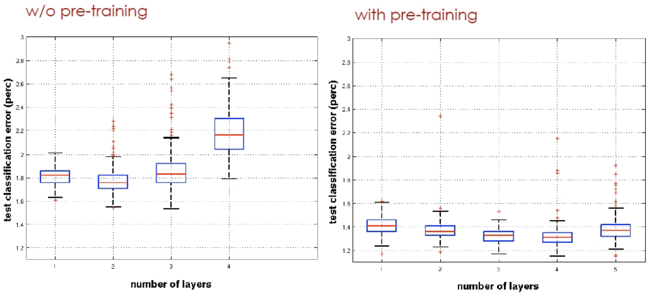

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- Regularization. Feature representations that are good for $P(x)$ are good for $P(y|x)$
- Optimization: Unsupervised pre-training leads to better regions of the space i.e. better than random initialization

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- De-noising Autoencoders: Input is corrupted by noise, but we attempt to reconstruct the uncorrupted image
- Contractive Autoencoders: The regularization term penalizes for the derivative:

$$\Omega(\mathbf{h}, \mathbf{x}) = \lambda \sum_i \|\nabla_{\mathbf{x}} \mathbf{h}_i\|_2^2$$

De-Noising Autoencoder: Intuition

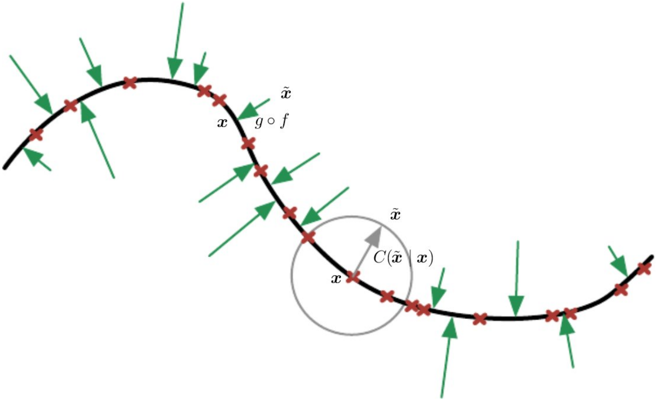
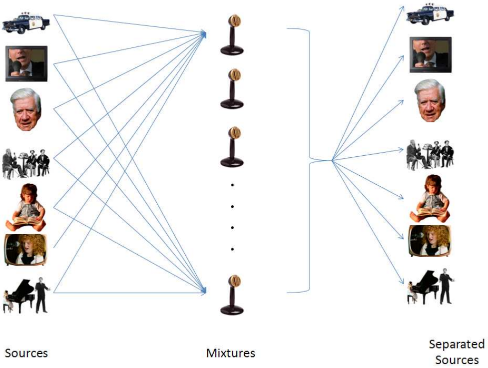


Figure: Goodfellow *et al.*

Another Linear Model: ICA

- Canonical example: Cocktail party problem



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- Here the bases are *independent* of each other

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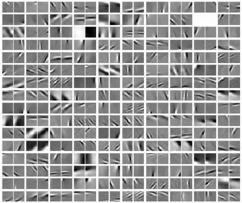
- In ICA $X = AH$ with A invertible
- PCA does compression, ICA doesn't do any compression $(p = d)$
- Some PCs are more important than others, not in the case with ICA

Filters



Short Digression: Distributed Representations

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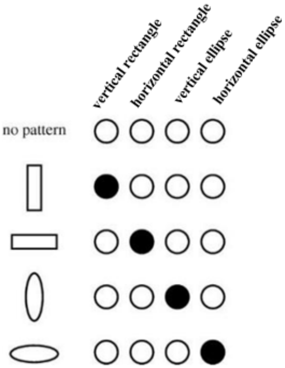
- One of the reasons of the power of Deep Networks are distributed representations (which unlike these toy cases are highly non-linear)
- What are distributed representations?

Distributed Representations: Intuition



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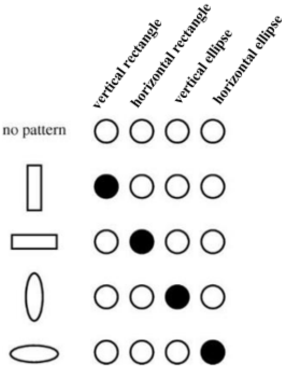
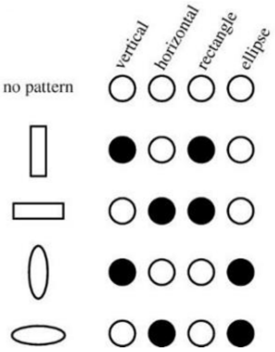
Distributed Representations: Intuition



- This is a *localist* representation: Every concept gets a code that has local structure
- Very easy to code, and easy to learn (mixture models build representations like this)

Distributed Representations: Intuition



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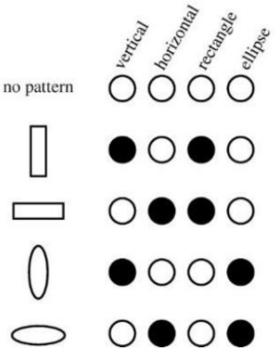

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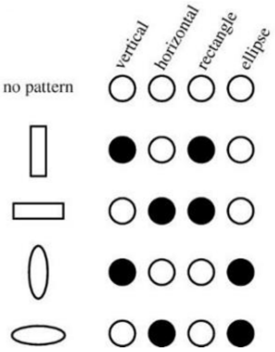

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- Given an exponential advantage in representational efficiency

Representations

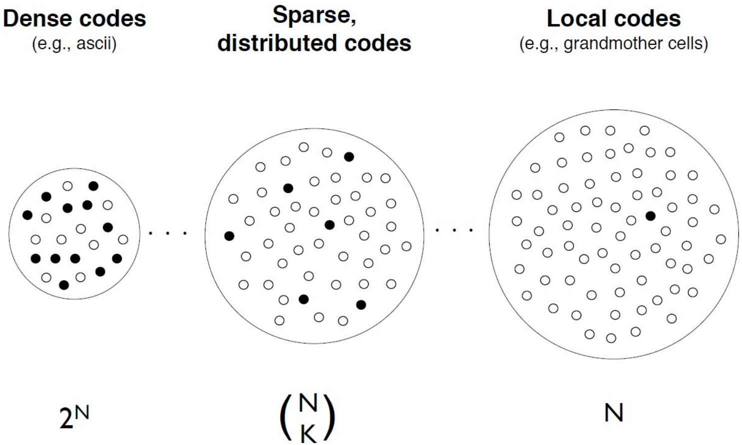
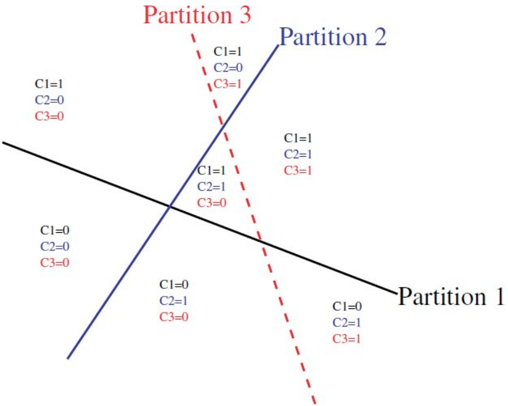


Figure: Bruno Olshausen

Representations



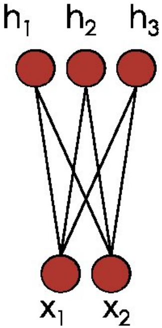


Figure: Yoshua Bengio (FTML Volume)

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    - $1 + m + \binom{m}{2} + \cdots + \binom{m}{d}$

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- Note: This is not specific to just unsupervised deep learning

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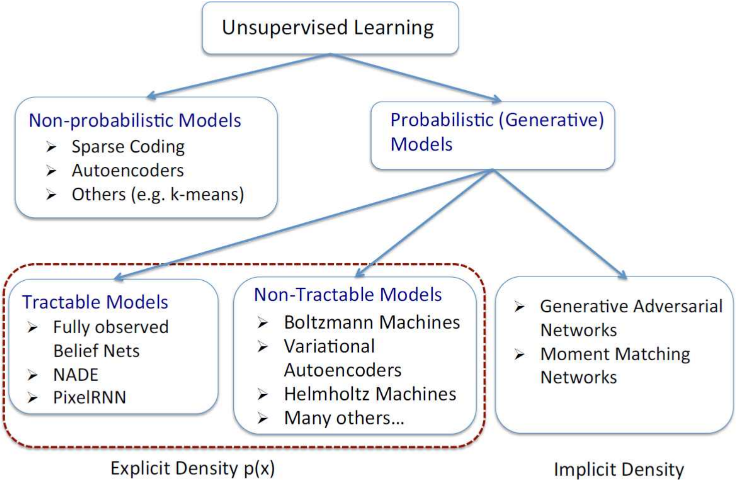

Figure: Ruslan Salakhutdinov

- To motivate Deep Neural Generative models, like before, let's seek inspiration from simple linear models first

Linear Factor Model

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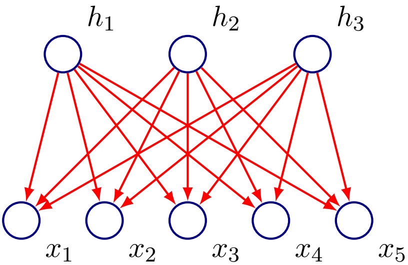
- Then: $\mathbf{x} = W\mathbf{h} + \mathbf{b} + \epsilon$

Linear Factor Model

- $P(\mathbf{h})$ is a factorial distribution



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- How do learn in such a model?
- Let's look at a simple example

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- Gives a simple generative model for the data; can draw samples!

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- Approaches PCA as $\sigma \to 0$

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- Estimation and inference can get complicated!
- Let's look at an approach to write these problems in a general form

Energy Based Models

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- Energies are in the log-probability domain:

$$\text{Energy}(\mathbf{x}) = \log \frac{1}{(ZP(\mathbf{x}))}$$

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- How do we specify the energy function?

Product of Experts Formulation

- In this formulation, the energy function is:

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- Contrast this with mixture models

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- Free Energy is just a marginalization of energies in the log-domain:

$$\text{FreeEnergy}(\mathbf{x}) = -\log \sum_{\mathbf{h}} \exp^{-(\text{Energy}(\mathbf{x},\mathbf{h}))}$$